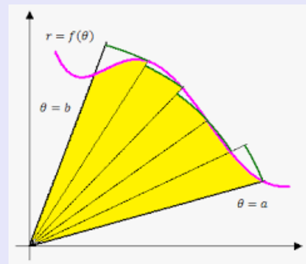
Calculus II

Lecture 21



Feb 19-8:47 AM

Class Qz 18

Find the $\boxed{\text{exact}}$ arc length of the parametric equation $x = e^t + e^{-t}$, $y = 3 - 2t$ for $0 \le t \le 1$ by using $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.

$$L = \int_0^1 \sqrt{(e^t - e^{-t})^2 + (-2)^2}\, dt$$

$$= \int_0^1 \sqrt{e^{2t} - 2 + e^{-2t} + 4}\, dt$$

$$= \int_0^1 \sqrt{e^{2t} + 2 + e^{-2t}}\, dt = \int_0^1 \sqrt{(e^t + e^{-t})^2}\, dt$$

$$= \int_0^1 (e^t + e^{-t})\, dt = (e^t - e^{-t})\Big|_0^1$$

$$= (e^1 - e^{-1}) - (\cancel{e^0}^{\,1} - \cancel{e^0}^{\,1}) = \boxed{e - \frac{1}{e}}$$

Jul 17-7:20 AM

Testing Series:

1) Integral Test

If $f(x)$ is cont., decreasing and Positive

If $a_n = f(n)$

then if $\int_1^\infty f(x)dx$ converges, then $\sum_{n=1}^\infty a_n$ Converges.

if $\int_1^\infty f(x)dx$ diverges, then $\sum_{n=1}^\infty a_n$ diverges.

Jul 17-8:22 AM

Test for Convergence

$\sum_{n=1}^\infty \frac{1}{n^4}$

Let $f(x) = \frac{1}{x^4}$

$f(x) > 0$, $f(x)$ is decreasing.

$f'(x) = \frac{-4}{x^5} < 0$

$f(x)$ is Cont. on $[1,\infty)$

$a_n = f(n) = \frac{1}{n^4}$

$$\int_1^\infty f(x)dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^4}dx$$

$$= \lim_{t\to\infty} \frac{x^{-3}}{-3}\Big|_1^t = \frac{-1}{3}\lim_{t\to\infty}\left(t^{-3} - 1^{-3}\right)$$

$$= \frac{-1}{3}\lim_{t\to\infty}\left[\cancel{\frac{1}{t^3}}^{\,0} - 1\right] = \boxed{\frac{1}{3}}$$

Since $\int_1^\infty \frac{1}{x^4}dx$ converges, then by I.T.

$\sum_{n=1}^\infty \frac{1}{n^4}$ Converges

Jul 17-8:26 AM

Test for Convergence

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

$f(x) = \dfrac{x}{x^2+1}$

$f(x) > 0 \quad [1,\infty)$

$f(x)$ is cont. $[1,\infty)$

$f'(x) = \dfrac{1(x^2+1) - x\cdot 2x}{(x^2+1)^2}$

$= \dfrac{1-x^2}{(x^2+1)^2} \quad (1,\infty)$

$f'(x) < 0$

$f(x)$ is decreasing

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx$$

$$= \lim_{t\to\infty} \int_1^t \frac{x}{x^2+1}\,dx$$

$u = x^2+1$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$$= \lim_{t\to\infty} \frac{1}{2}\int_2^{t^2+1} \frac{1}{u}\,du = \frac{1}{2}\lim_{t\to\infty} \ln u \Big|_2^{t^2+1}$$

$$= \frac{1}{2}\lim_{t\to\infty} \left[\ln(t^2+1) - \ln 2\right] \quad \infty$$

$$= \frac{1}{2}\lim_{t\to\infty} \ln \frac{t^2+1}{2} = \infty$$

$\int_1^{\infty} \frac{x}{x^2+1}\,dx$ diverges

By I.T,

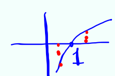
$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges.

Jul 17-8:32 AM

Test the Series

$$\sum_{n=2}^{\infty} \frac{1}{n\ln n}$$

$f(x) = \dfrac{1}{x\ln x}$



$\ln 1 = 0$

$f(x) > 0$ for $[2,\infty)$

$f(x)$ is cont. $[2,\infty)$

$f(x) = (x\ln x)^{-1}$

$f'(x) = -1\,(x\ln x)^{-2}\cdot\left(1\cdot\ln x + x\cdot\frac{1}{x}\right)$

$f'(x) = \dfrac{-(1+\ln x)}{(x\ln x)^2} < 0$

$f(x)$ is decreasing

$$\int_2^{\infty} \frac{1}{x\ln x}\,dx$$

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{u}\,du = \ln u + C = \ln(\ln x)$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$\int_2^{\infty} \frac{1}{x\ln x}\,dx = \lim_{t\to\infty} \ln(\ln x)\Big|_2^t$$

$$= \lim_{t\to\infty} \left[\ln(\ln t) - \ln 2\right] = \infty$$

$\int_2^{\infty} \frac{1}{x\ln x}\,dx$ diverges

by I.T.,

$\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ diverges

Jul 17-8:41 AM

Test the Series

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$$

$f(x) = \frac{\ln x}{x^3}$

$f(x) > 0 \quad (1, \infty)$

$f(x)$ is Cont. $[1, \infty)$

$$f'(x) = \frac{\frac{1}{x}\cdot x^3 - \ln x \cdot 3x^2}{(x^3)^2} = \frac{x^2 - 3x^2\ln x}{x^6} = \frac{1 - 3\ln x}{x^4} < 0$$

$f(x)$ is decreasing

$$\int_1^{\infty} \frac{\ln x}{x^3}\,dx$$

$$\int \frac{\ln x}{x^3}\,dx = \int x^{-3}\ln x\,dx$$

$u = \ln x \quad dv = x^{-3}dx$

$du = \frac{1}{x}dx \quad v = \frac{x^{-2}}{-2}$

$$= \frac{-1}{2}x^{-2}\ln x + \frac{1}{2}\int \frac{1}{x}\cdot x^{-2}dx$$

$$= \frac{-1}{2}x^{-2}\ln x + \frac{1}{2}\int x^{-3}dx = \frac{-1}{2}\left[\frac{\ln x}{x^2} - \frac{x^{-2}}{-2}\right]$$

$$= \frac{-1}{2}\left[\frac{\ln x}{x^2} + \frac{1}{2x^2}\right]$$

$$\int_1^{\infty} \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty}\left[\frac{-1}{2}\left[\cancel{\frac{\ln t}{t^2}}^{\,0} + \cancel{\frac{1}{2t^2}}^{\,0} - \cancel{\frac{\ln 1}{1^2}}^{\,0} - \frac{1}{2\cdot 1^2}\right]\right]$$

$$= \frac{-1}{2}\cdot\frac{-1}{2} = \boxed{\frac{1}{4}}$$

Converges by I.T., Series Converges.

Jul 17-8:54 AM

Comparison Test:

$\sum a_n$ & $\sum b_n$, $a_n > 0$, $b_n > 0$

If $a_n < b_n$ and $\sum b_n$ converges then $\sum a_n$ converges.

If $a_n > b_n$ and $\sum b_n$ diverges then $\sum a_n$ diverges.

Test for Convergence.

$$\sum_{n=1}^{\infty} \frac{n}{n^3+1}$$

a_n

$\approx \frac{n}{n^3} = \frac{1}{n^2}$

Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

b_n

$a_n < b_n$? ✓

$$\frac{n}{n^3+1} < \frac{1}{n^2}$$

$$n\cdot n^2 < n^3+1$$

$$n^3 < n^3+1$$

Work on $\sum_{n=1}^{\infty} \frac{1}{n^2}$

use I.T. or P-Series

Converges

by Comparison Test $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ Converges

Jul 17-9:07 AM

Test for Convergence

$$\sum_{n=1}^{\infty} \frac{n+2}{n\sqrt{n}} \rightarrow \approx \frac{n}{n\sqrt{n}} = \frac{1}{\sqrt{n}}$$

Compare with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Is $a_n > b_n$? ✓

$$\frac{n+2}{n\sqrt{n}} > \frac{1}{\sqrt{n}}$$

$$\sqrt{n}(n+2) > n\sqrt{n} \cdot 1$$

$$\cancel{n\sqrt{n}} + 2\sqrt{n} > \cancel{n\sqrt{n}}$$

$$2\sqrt{n} > 0$$

by Comparison Test $\sum_{n=1}^{\infty} \frac{n+2}{n\sqrt{n}}$ diverges

Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges

I.T. or

P-Series Test

Jul 17-9:16 AM

Test the Series

$$\sum_{n=1}^{\infty} \underbrace{\frac{1}{\sqrt{n^2+1}}}_{a_n} \rightarrow \approx \frac{1}{\sqrt{n^2}} = \frac{1}{n}$$

Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$

Diverges by I.T., P-Series

Harmonic Series

$a_n > b_n$?

$$\frac{1}{\sqrt{n^2+1}} > \frac{1}{n}$$

$$\frac{1}{\sqrt{n^2+1}} > \frac{1}{\sqrt{n^2}}$$

$$\sqrt{n^2} > \sqrt{n^2+1}$$

NO

Jul 17-9:21 AM

Limit Comparison Test:

$\sum a_n$ & $\sum b_n$ $\quad a_n > 0$, $b_n > 0$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L, \quad L > 0, \quad L \text{ is a number}$$

Either both Converge or both diverges

From Last example

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\sqrt{\frac{n^2}{n^2}+\frac{1}{n^2}}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1$$

So $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges

by L.C.T.

Jul 17-9:27 AM

Test the Series

$$\sum_{n=1}^{\infty} \frac{1}{3^n+1} \approx \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$$

Compare with $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

Converges by I.G.S. $r = \frac{1}{3} < 1$

Use limit Comparison Test

$$\lim_{n\to\infty} \frac{\frac{1}{3^n+1}}{\frac{1}{3^n}} = \lim_{n\to\infty} \frac{3^n}{3^n+1}$$

$$= \lim_{n\to\infty} \frac{1}{1+\cancel{\frac{1}{3^n}}\to 0} = 1$$

by L.C.T.

both converge

Jul 17-9:33 AM

Test the Series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n^2} \rightarrow \approx \frac{\sqrt{n^4}}{n^3} = \frac{n^2}{n^3} = \frac{1}{n}$$

Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$

Diverges

Harmonic Series, P-Series $P=1$

I.T.

Limit Comparison Test

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n^4+1}}{n^3+n^2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n\sqrt{n^4+1}}{n^3+n^2}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^4+1}}{n^2+n} = \lim_{n\to\infty} \frac{\sqrt{\frac{n^4}{n^4} + \frac{1}{n^4}}}{\frac{n^2}{n^2} + \frac{n}{n^2}} = 1$$

by L.C.T.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n^2}$$

Diverges.

both diverge.

Jul 17-10:01 AM

Test the Series

$$\sum_{n=1}^{\infty} \sin\frac{1}{n}$$

Compare $\sum_{n=1}^{\infty} \frac{1}{n}$

Diverges by ...

Divergence Test

$$\lim_{n\to\infty} a_n$$

Does not help us for convergence.

$$= \lim_{n\to\infty} \sin\frac{1}{n} = \sin 0 = 0$$

we cannot use divergence Test.

$h = \frac{1}{n}$

Use L.C.T.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim_{h\to 0} \frac{\sin h}{h} = 1$$

by L.C.T.

$\sum_{n=1}^{\infty} \sin\frac{1}{n}$ Diverges

because $\sum_{n=1}^{\infty} \frac{1}{n}$ Div.

Jul 17-10:08 AM

Test the Series

$$\sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^{n} \leftarrow \text{Power}$$

Root Test

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{2n+1}\right)^{n}} = \lim_{n\to\infty}\frac{n}{2n+1} = \frac{1}{2} < 1$$

Converges

Ratio Test

$$\lim_{n\to\infty}\left|\frac{\left(\frac{n+1}{2(n+1)+1}\right)^{n+1}}{\left(\frac{n}{2n+1}\right)^{n}}\right|$$

$$= \lim_{n\to\infty}\frac{\left(\frac{n+1}{2n+3}\right)^{n}\cdot\frac{n+1}{2n+3}}{\left(\frac{n}{2n+1}\right)^{n}}$$

$$= \lim_{n\to\infty}\left(\frac{\frac{n+1}{2n+3}}{\frac{n}{2n+1}}\right)^{n}\cdot \lim_{n\to\infty}\frac{n+1}{2n+3} \rightarrow \tfrac{1}{2}$$

$$= \lim_{n\to\infty}\left(\frac{(2n+1)(n+1)}{n(2n+3)}\right)^{n}\cdot\frac{1}{2}$$

$$= \boxed{\lim_{n\to\infty}\left(\frac{2n^{2}+\cdots}{2n^{2}+\cdots}\right)^{n}}\cdot\frac{1}{2}$$

$1 \qquad 1^{\infty}$

Jul 17-10:15 AM

Test the Series

$$\sum_{n=1}^{\infty}\frac{1}{3^{n}\cdot n!}$$

Ratio test

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{1}{3^{n+1}\cdot(n+1)!}}{\frac{1}{3^{n}\cdot n!}} = \lim_{n\to\infty}\frac{\cancel{3^{n}}\cdot\cancel{n!}}{\cancel{3^{n+1}}\cdot\cancel{(n+1)!}}$$

$$= \lim_{n\to\infty}\frac{1\cdot 1}{3\cdot(n+1)} = \frac{1}{3}\lim_{n\to\infty}\frac{1}{n+1}$$

$$= \frac{1}{3}\cdot 0 = 0$$

Converges by Ratio Test.

Jul 17-10:25 AM

Alternating Series Test:

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

✓ 1) $0 < a_{n+1} \leq a_n$

✓ 2) $\lim_{n\to\infty} a_n = 0$

$\Rightarrow$ Converges by A.S.T.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \qquad a_n = \frac{1}{n}$$

✓ $a_{n+1} \leq a_n$? $\quad \frac{1}{n+1} \leq \frac{1}{n}$

$n \leq n+1$

Converges by A.S.T.

✓ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$

Jul 17-10:29 AM

Test the Series

$$\sum_{n=1}^{\infty} (-1)^n \frac{\tan^{-1} n}{n^2}$$

$a_n = \frac{\tan^{-1} n}{n^2}$

we have an alternating Series

1) $0 < a_{n+1} \leq a_n$

$$\frac{\tan^{-1}(n+1)}{(n+1)^2} \leq \frac{\tan^{-1} n}{n^2}$$

$\square > \boxed{\text{Next}}$

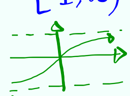
Decreasing

$$f(x) = \frac{\tan^{-1} x}{x^2}$$

$$f'(x) = \frac{\frac{1}{x^2+1} \cdot x^2 - \tan^{-1} x \cdot 2x}{(x^2)^2} = \frac{x\left[x - 2(x^2+1)\cdot \tan^{-1} x\right]}{x^4(x^2+1)}$$

$$= \frac{x - 2(x^2+1)\tan^{-1} x}{x^3(x^2+1)} = \frac{-}{+}$$

$[1, \infty)$



$f'(x) < 0 \rightarrow f(x)$ decreasing

2) $\lim_{n\to\infty} a_n = \lim_{x\to\infty} \frac{\tan^{-1} x}{x^2} = \frac{\frac{\pi}{2}}{\infty} = 0$

Our Series Converges by A.S.T.

Jul 17-10:35 AM

Test the Series

$$\sum_{n=1}^{\infty} \frac{2^{n^2}}{n!}$$

Hint: when working with factorial $\rightarrow$ use Ratio Test.

$$\lim_{n\to\infty} \frac{\dfrac{2^{(n+1)^2}}{(n+1)!}}{\dfrac{2^{n^2}}{n!}} = \lim_{n\to\infty} \frac{n!\cdot 2^{n^2+2n+1}}{(n+1)!\cdot 2^{n^2}}$$

a_{n+1} , a_n

$$= \lim_{n\to\infty} \frac{2^{2n+1}}{n+1}$$

$$= \infty > 1 \quad \underline{\text{Diverges}}$$

Jul 17-10:46 AM

Find all values of x for which the series $\sum_{n=1}^{\infty} n!(2x-1)^n$ converges.

Ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(n+1)!}(2x-1)^{n+1}}{\cancel{n!}(2x-1)^n}\right|$$

$$= \lim_{n\to\infty} |(n+1)\cdot(2x-1)| < 1$$

$$= |2x-1|\cdot\lim_{n\to\infty}(n+1) < 1$$

$2x-1=0$

$x=\frac{1}{2}$

Jul 17-10:54 AM

For what values of x the Series

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$$ Converge?

Hint $\rightarrow$ Use Root Test

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n x^n}{n^2}\right|} = \lim_{n\to\infty} \frac{|x|}{\sqrt[n]{n^2}} = \frac{|x|}{1} = |x|$$

$$\lim_{n\to\infty} \sqrt[n]{n^2} = \lim_{n\to\infty} n^{\frac{2}{n}} = \infty^0$$

$$y = n^{\frac{2}{n}}$$

$$\ln y = \ln n^{\frac{2}{n}}$$

$$\ln y = \frac{2\ln n}{n}$$

$$\lim_{n\to\infty} \frac{2\ln n}{n} = 2\cdot 0$$

$$= e^0 = 1$$

$$|x| < 1 \rightarrow \boxed{-1 < x < 1}$$

Jul 17-11:01 AM